IEEE Pacific Rim Conference on Communications, Computers and Signal Processing
June 1st - 2nd, 1989

PHASE SMOOTHING FUNCTIONS FOR FULL RESPONSE CPM

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Abstract - Continuous Phase Modulation (CPM) is examined for low complexity receivers, i.e., those that operate on only one symbol interval. The tradeoff between the bandwidth and bit error rate (BER) performance of these simple systems is studied. The phase smoothing functions that minimize the effective bandwidth for a given BER are calculated. A linear and a polynomial approximation to these phase smoothing functions are chosen. The three families are compared using their bandwidth – BER performance tradeoffs, power spectral densities, out–of–band powers and bandwidth efficiencies. It is shown that CPM schemes with small modulation indices ($h \leq 0.5$) offer good tradeoffs between bandwidth and BER performance.

1. Introduction

The binary signaling scheme that gives the best bit error rate (BER) performance with one symbol interval demodulation is obtained when antipodal signals are used. For carrier modulation, Binary Phase Shift Keying (BPSK) is the most commonly used scheme. However, when spectral efficiency is important, BPSK is not recommended. There are a large number of modulation schemes that are spectrally more efficient than BPSK. This work considers one such class, namely Continuous Phase Modulation (CPM).

CPM signals with good spectral properties and BER performance have been found. Past research effort has emphasized signal design for optimal receivers. However, optimal receivers for CPM are complex; they consist of a bank of correlators whose outputs are sampled and fed into a Viterbi processor [1, pp.246-252] The complexity of the optimal receiver has limited the usefulness of CPM schemes to the point where only MSK–like and tamed FM (TFM) systems have been implemented.

In this paper we take a different approach. Signals are designed with simpler receivers in mind, i.e., receivers that base their decision on only one symbol interval. This results in the familiar binary correlation receiver [2, pp.84-86] Although this approach reduces the complexity of the receiver, there is a corresponding loss in BER performance.

With these simple receivers in mind, we investigate the design and properties of CPM signals.

2. CPM Definitions and Performance Measures

A CPM signal can be represented by its complex envelope as,

$$\tilde{s}(t) = e^{j\phi(t)} \quad -\infty < t < \infty$$

where the phase function $\phi(t)$ is given by (T = symbol interval),

$$\phi(t) = 2\pi h \sum_{n=-\infty}^{\infty} a(n) q[t-(n-1)T] \quad -\infty < t < \infty$$

The phase function depends on the information sequence, $a(n)$, which takes on the values ± 1, and the phase smoothing function, $q(t)$, which can be any continuous function satisfying,

$$q(t) = \begin{cases} 0 & t \leq 0 \\ 0.5 & t \geq T \end{cases}$$

For our investigation, we are interested in only one symbol interval. The phase function is then given by,

$$\phi(t) = \phi(0) + 2\pi h a(n) q(t) \quad 0 \leq t \leq T$$

In one symbol interval, only two different signals are possible. Namely, the signals that correspond to the phase functions,

$$\phi_1(t) = \phi(0) + 2\pi h q(t) \quad 0 \leq t \leq T$$
$$\phi_2(t) = \phi(0) - 2\pi h q(t) \quad 0 \leq t \leq T$$

The complex envelopes of these signals are then given by,

$$\tilde{s}_1 = e^{j\phi_1(t)} \quad \tilde{s}_2 = e^{j\phi_2(t)} \tag{1}$$

The BER performance of a binary signaling scheme can be completely described by a normalized correlation coefficient, $\tilde{\rho}$, defined as,

$$\tilde{\rho} = \frac{< \tilde{s}_1, \tilde{s}_2 >}{||\tilde{s}_1|| \, ||\tilde{s}_2||}$$

This represents the inner product of the two signals divided by the norm of each of the signals. Note that from the Schwartz inequality, $|\tilde{\rho}| \leq 1$.

For the signals defined in (1), $\tilde{\rho}$ is then given by,

$$\tilde{\rho} = \frac{1}{T}\int_0^T e^{j4\pi h q(t)}\, dt$$

For coherent detection, only the real part of $\tilde{\rho}$ is important.

$$\rho = \Re[\tilde{\rho}] = \frac{1}{T}\int_0^T \cos[4\pi h q(t)]\, dt \tag{2}$$

Note that :$-1 \leq \rho \leq 1$.

The other performance measure of interest is bandwidth. There are many ways to describe a signal's bandwidth, but the one we shall use is the effective bandwidth defined by,

$$B_e^2 = \frac{4\int_0^\infty (f-f_0)^2 S_+(f)\, df}{\int_0^\infty S_+(f)\, df}$$

where $S_+(f)$ is the positive frequency part of the signal's spectrum. This was shown by [3] to be approximately given by,

$$B_e^2 = \frac{4h^2}{T}\int_0^T \dot{q}^2(t)\, dt \tag{3}$$

Note that : $B_e T \geq h$.[1] Now, both performance measures of interest are functions of the phase smoothing function, $q(t)$. The ideal $q(t)$ is one that minimizes ρ and B_e simultaneously. However, as we will show, this is not possible and therefore a tradeoff emerges.

The best BER performance, i.e., when $\rho = -1$, cannot be realized for the CPM signals under consideration[2]. Only for discontinuous $q(t)$ can $\rho = -1$ be obtained. This type of signal is shown as f_3 in figure 2.1. Note that for f_3, $B_e = \infty$.

[1] See Appendix B for details

[2] Refer to Appendix A for details and a proof.

As for B_e, it was shown in Appendix B that its minimum is attained for the linear phase smoothing function, f_1, shown in figure 2.1. For f_1, $\rho = \text{sinc}(2h)$ which at best is approximately -0.2.

We see then that when ρ is minimized, B_e is maximized and when B_e is minimized, ρ is relatively high. Since both cannot be minimized simultaneously, a tradeoff exists. A heuristic compromise is shown as f_2 in figure 2.1, where the three phase smoothing functions in form an increasing sequence in B_e, but a decreasing sequence in ρ.

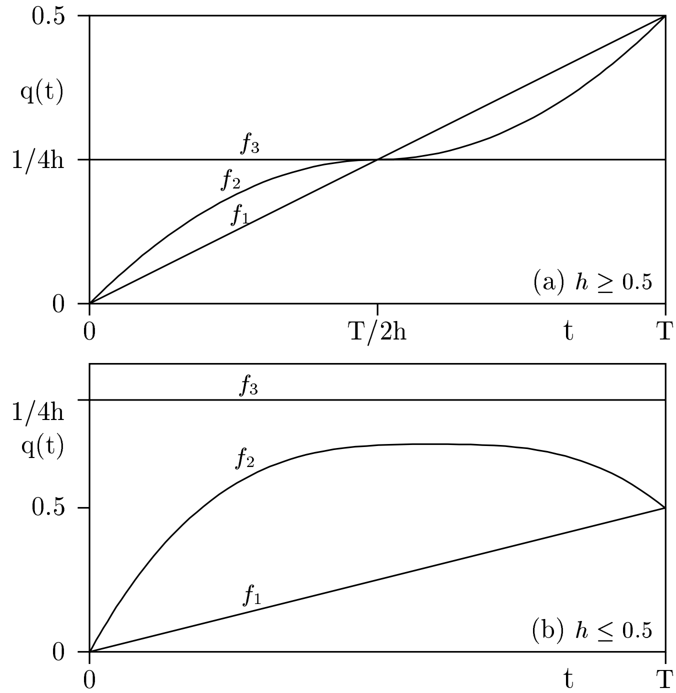

Figure 2.1: Heuristic Phase Smoothing Functions

The next question that comes to mind is how small can ρ be made for a given value of B_e? This can be answered by using calculus of variations.

Basically we want to minimize (2) with (3) equal to a constant. This is a standard variational problem [4, pp.42-46] and reduces to solving the differential equation,

$$\ddot{q}(t) = \frac{-\pi}{2h\lambda} \sin[4\pi h q(t)] \qquad (4)$$

with the boundary conditions: $q(0) = 0$ and $q(T) = 0.5$. The parameter λ is chosen so that B_e is the desired value.

By examining the differential equation, we see that the solution to equation (4) must have a form similar to f_2 in figure 2.1, i.e., concave down for $q(t) < 1/(4h)$ and concave up for $q(t) > 1/(4h)$. Numerical solution of (4) leads to phase smoothing functions such as those in figure 2.2.

It should be noted that although the tradeoff has been optimized, this may not lead to optimal tradeoffs when some other measure of bandwidth is used. However, a low B_e should lead to "good" spectral performance.

3. Families of Phase Smoothing Functions

In this paper, we look at three families of phase smoothing functions for comparison. The first one will be called OPT and is the family of functions found to optimize the $\rho - B_e$ tradeoff. The second is a piecewise linear approximation to OPT and will be referred to as PWL. The third is a polynomial approximation called POLY.

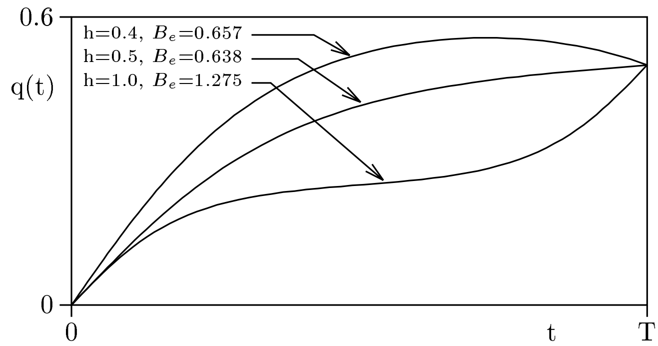

Figure 2.2: OPT Phase Smoothing Functions $\rho = -0.5$

PWL is completely determined by a parameter, A, as follows,

$$h \geq 0.5 : q(t) = \begin{cases} (At)/(4hT) & 0 \leq t \leq T/A \\ 1/(4h) & T/A < t \leq B \\ [A/(4hT)](t-B) + 1/(4h) & B < t \leq T \end{cases}$$

$$h < 0.5 : q(t) = \begin{cases} (Adt)/T & 0 \leq t \leq T/A \\ d & T/A < t \leq B \\ (-Ad/T)(t-B) + d & B < t \leq T \end{cases}$$

The parameter, B is chosen so that $q(T) = 0.5$. Typical members of PWL are shown in figure 3.1.

For POLY, the parameter A is the degree of the polynomial approximation.

$$h \geq 0.5 : q(t) = \begin{cases} 1/(4h) - k_1[T/(2h) - t]^A & 0 \leq t \leq T/(2h) \\ k_2[t - T/(2h)]^A + 1/(4h) & T/(2h) < t \leq T \end{cases}$$

$$h < 0.5 : q(t) = \begin{cases} d - k(B-t)^A & 0 \leq t \leq B \\ d - k(t-B)^A & B < t \leq T \end{cases}$$

The constants B, k_1, k_2, and k are chosen so that $q(0) = 0$ and $q(T) = 0.5$. Typical POLY phase smoothing functions are shown in figure 3.2.

For both families, d is chosen so that when $A = 1$, $d = 0.5$ and when $A = \infty$, $d = 1/4h$.

$$d = (1 - 1/A)[1/(4h) - 1/2] + 1/2$$

Notice that MSK is obtained for both families when $h = 0.5$ and $A = 1$. PSK is obtained when $A = \infty$.

Clearly, the choices for d in PWL and POLY are not unique and were chosen for their simplicity. Another choice may give a better approximation to OPT, but that is of lesser importance here.

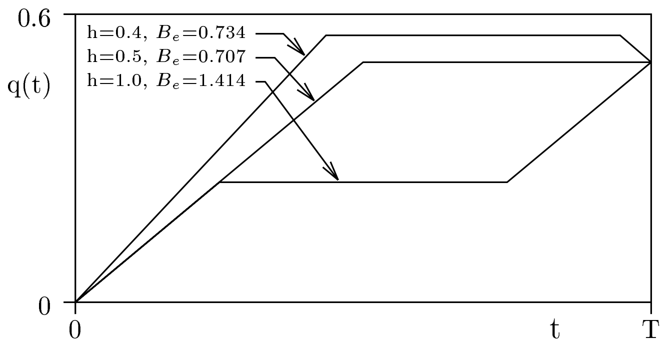

Figure 3.1: PWL Phase Smoothing Functions $\rho = -0.5$

4. Performance Results

The tradeoff between ρ and B_e for OPT is shown in figure 4.1 for various values of h. Notice that there are phase smoothing functions that have the same ρ as MSK, but a lower effective bandwidth. Another interesting feature is that for $h < 0.5$, the curve for h lies below the curve for $1 - h$, e.g., the curve for $h = 0.2$ lies below the one for $h = 0.8$. It can also be seen that

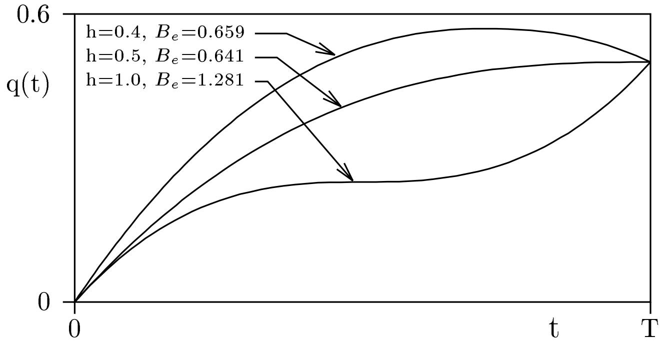

Figure 3.2: POLY Phase Smoothing Functions $\rho = -0.5$

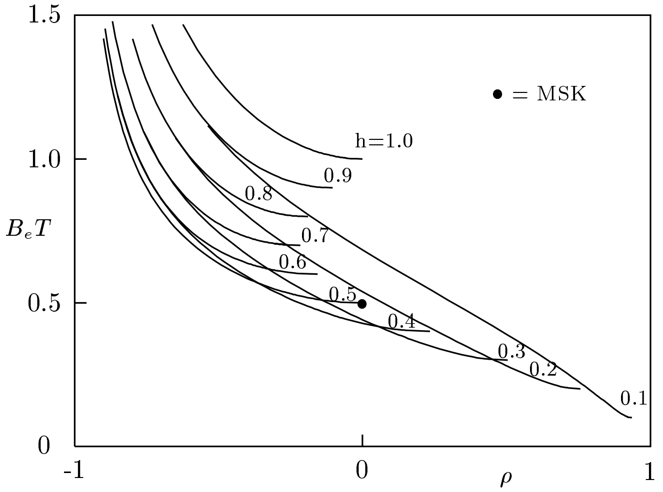

Figure 4.1: OPT $B_e - \rho$ Tradeoffs for Various h

for lower values of ρ, $h = 0.5$ gives the best tradeoff. This can be attributed to the fact that $1/4h$ is equal to 0.5 and therefore results in a "smoother" function at $t = T$.

The curves for PWL and POLY were calculated and showed similar B_e vs. ρ tradeoffs. As was the case for OPT, $h = 0.5$ gave the best tradeoff for low values of ρ. Hence, we will restrict our performance results to those involving $h = 0.5$ and $\rho \leq -0.5$.

The tradeoffs for the three families are compared in figure 4.2 for $h = 0.5$. This shows that POLY gives almost the same tradeoff as OPT, while PWL is slightly inferior.

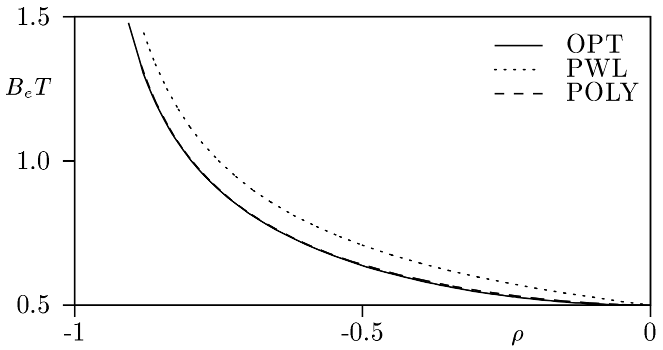

Figure 4.2: $B_e - \rho$ Tradeoff Comparison ($h = 0.5$)

The power spectral densities were calculated using the method in [1, pp. 149-154] They are plotted for OPT in figure 4.3 along with those of PSK and MSK for comparison. From this figure, we can see that as ρ decreases, the main lobe width increases and the rolloff becomes less rapid. Plots of the other two families show similar results.

The three families are compared for $\rho = -0.5$ in figure 4.4.

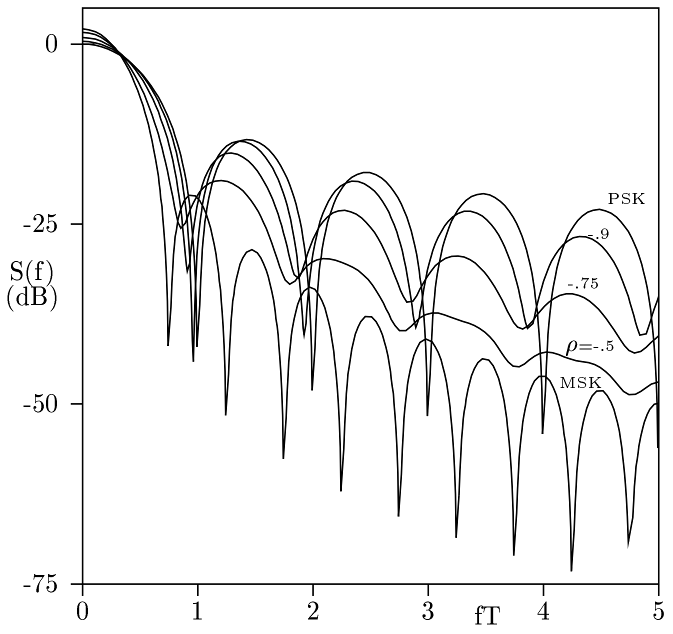

Figure 4.3: OPT Power Spectral Densities ($h = 0.5$)

This shows that the close similarity between the effective bandwidths of OPT and POLY has translated into similar power spectra. PWL, on the other hand, appears slightly inferior in main lobe width and spectral rolloff.

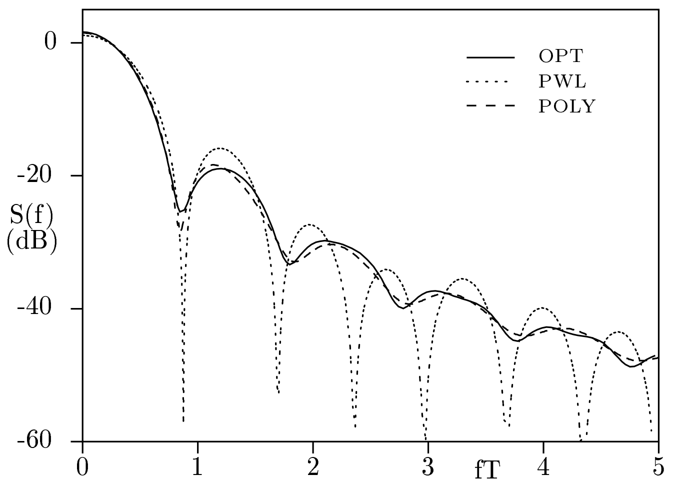

Figure 4.4: Power Spectrum Comparison ($h = 0.5, \rho = -0.5$)

Out of band power was also computed and similarly shows that as ρ decreases, the rolloff becomes less rapid. This is shown in figure 4.5 for OPT.

The three families are compared for $\rho = -0.5$ in figure 4.6. Again, OPT and POLY are quite close. However, while PWL is inferior in a region close to the main lobe, it becomes comparable to the other two farther away.

Performance was also evaluated on the "Efficiency Plane" [5] shown in figure 4.7. This allows a comparison of the ρ–bandwidth tradeoff with the limit given by Shannon. The bandwidth measure used was 99% power. From this figure, we can see that OPT and POLY are close while PWL is slightly inferior.

5. Conclusions

For the simple receiver considered in this paper, we have shown that the BER performance of antipodal signaling cannot be obtained. We also found that BER performance and bandwidth cannot be minimized simultaneously and so a tradeoff

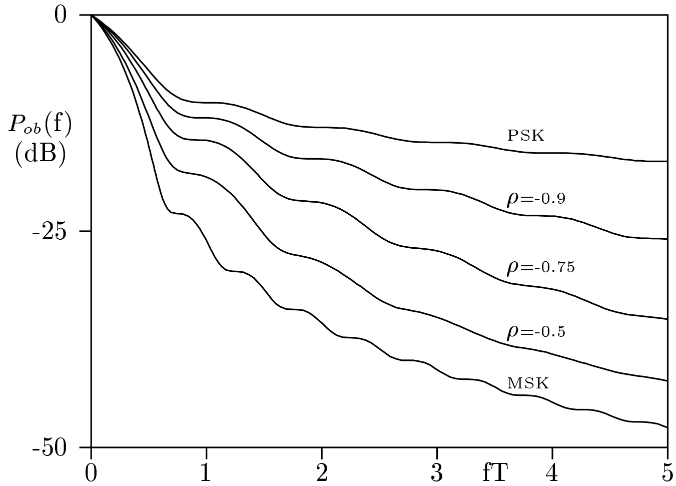

Figure 4.5: OPT Out-of-Band Power Plots (h=0.5)

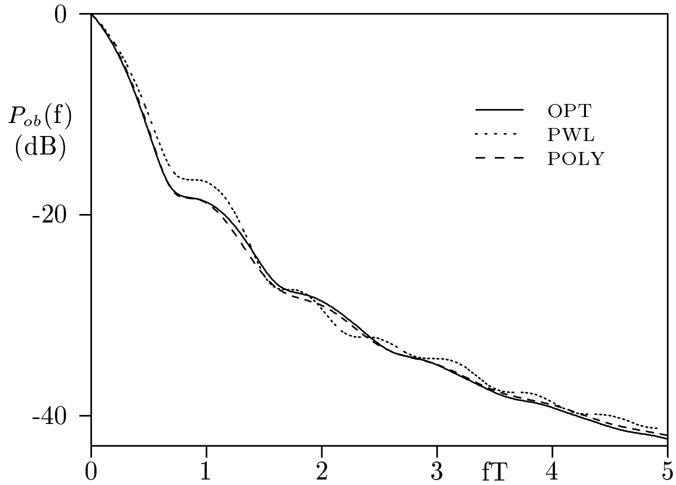

Figure 4.6: Out-of-Band Power Comparison (h=0.5,ρ=-0.5)

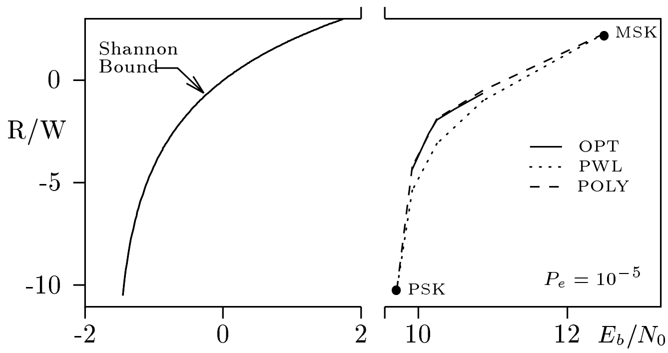

Figure 4.7: Bandwidth Efficiency Plane ($h = 0.5$)

emerges. Modulation indices greater than 0.5 offer no B_e vs. ρ tradeoff advantage over values of h less than or equal to 0.5.

The optimal phase smoothing functions can be approximated well by a polynomial. Smaller values of effective bandwidth generally lead to better spectral performance and is therefore a useful bandwidth measure. However, optimization using effective bandwidth is not equivalent to optimization with respect to other bandwidth measures.

The general shape of phase smoothing functions that lead to good tradeoffs has been found. This, combined with the results of the comparisons, gives some guidelines for the design of phase smoothing functions. The one that would be used in practice depends on the performance criterion of most interest, i.e., bandwidth or bit error rate.

Appendix A

Theorem:

Consider the set of continuous functions in the interval [0,T] satisfying $f(0) = 0$ and $f(T) = 0.5$. Define the functional $\rho(f) = \rho[f(t)]$ as,

$$\rho(f) = \frac{1}{T}\int_0^T \cos[4\pi h f(t)]\, dt$$

where $h > 0$ is a constant. Then for any such function, $f(t)$, there exists another function, $g(t)$, such that $\rho(g) < \rho(f)$.

Proof:

Any function, $g(t)$, satisfying,

$$\begin{array}{ll} \cos[4\pi h g(t)] < \cos[4\pi h f(t)], & t \in [a,b], 0 \leq a < b \leq T \\ g(t) = f(t) & \text{otherwise} \end{array} \tag{5}$$

would be such a function. To show that such a function can always be found, consider a short interval [0,ϵ]. If $f(t)$ increases from 0, then choosing $g(t)$ to be larger than $f(t)$ but less than $1/(4h)$ will result in (5) above. Similarly, for $f(t)$ decreasing from 0, $g(t)$ should be less than $f(t)$ but larger than $-1/(4h)$.

This theorem tells us that $\rho = -1$ cannot be obtained for continuous phase-smoothing functions. If such a function existed, it would contradict the theorem because -1 is the smallest value ρ can take.

Appendix B

Problem:

Minimize,

$$B_e^2 = \frac{4h^2}{T}\int_0^T \dot{q}^2(t)\, dt \tag{6}$$

subject to,

$$\int_0^T \dot{q}(t)\, dt = 0.5$$

By inspection of the functionals, there is no bias given to any of the points of $\dot{q}(t)$, i.e., all points are treated equally. Therefore, one would expect that the solution to the above problem would be $\dot{q} = \textit{constant}$ in [0,T]. More formally, variational methods (see, for example, [4, pp.42-46]) can be used to show that the solution must satisfy, $\ddot{q} = 0$. Thus leading to the expected solution, $\dot{q}(t) = 1/(2T)$. Substituting this solution into (6), we see that $B_e T = h$.

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