Correspondence

Optimization of Coded GMSK Systems

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Abstract—**In this correspondence, block and convolutional coding for Gaussian minimum shift keying (GMSK) in additive white Gaussian noise (AWGN) channels is investigated. The performance improvement that can be obtained without an increase in bandwidth is found for block and convolutional codes with various block lengths and constraint lengths, respectively, when a simple suboptimal demodulation scheme is used. For a fixed system transmission bandwidth, the tradeoff between the percentage of bandwidth to use for coding and the percentage to use for modulation is examined and optimized for various values of bandwidth.**

Index Terms—**Block codes, convolutional codes, encoding, Gaussian minimum shift keying (GMSK), optimization methods.**

I. Introduction

Gaussian minimum shift keying (GMSK) is a popular modulation method that was first proposed in 1981 [1]. It has since been adopted as the modulation scheme for cellular systems such as GSM and DCS1800 [2] due to such favorable properties as bandwidth efficiency and constant envelope. Although the performance of GMSK has been analyzed by several researchers, e.g., [3]-[7], coding for GMSK has received little attention. Some work, such as [8], has been done but optimization of coding for GMSK has not been examined fully.

It is well known that coding increases the required transmission bandwidth when considered independently of modulation. However, the best tradeoff between modulation and coding for a given transmission bandwidth is not well understood. In this correspondence, we examine this tradeoff for coded GMSK systems in additive white Gaussian noise (AWGN) channels and determine the optimal amount of coding for a range of transmission bandwidths. We also derive the amount of gain that can be achieved compared to an uncoded system that uses the same transmission bandwidth.

In order to keep the bandwidth of the coded system the same as that of the uncoded system, we use the coding method shown in Fig. 1. The encoder in the coded system increases the number of bits that must be transmitted. Therefore, the bit interval of the coded bits must be shortened in order to keep the information transmission rate constant. A shorter bit interval results in a larger transmission bandwidth. Therefore, the modulator used in the coded system must adopt a smaller value of B_b, the modulator's prefilter bandwidth which is defined in the next section, than that of the modulator used in the uncoded system in order to keep the overall transmission bandwidth fixed. Thus, uncoded and coded systems with the same overall bandwidth can be produced.

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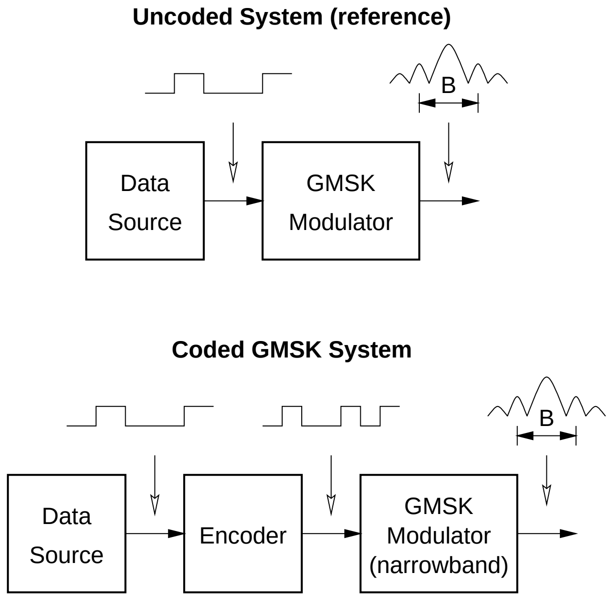

Fig. 1. The coded GMSK system examined in this correspondence compared to the uncoded system used as a reference.

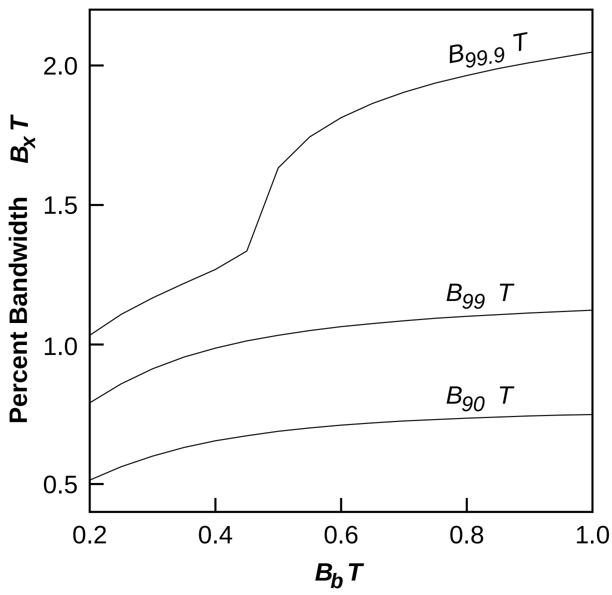

Fig. 2. Percent power containment bandwidths for GMSK.

The rest of the correspondence is organized as follows. First, we start with some preliminary discussion of GMSK and the performance measures necessary to evaluate the system. Second, we consider block codes and optimize the combination of coding and modulation. Third, we examine the optimization problem for convolutional codes and then finish with some conclusions.

II. Preliminaries

A. GMSK Modulation

A GMSK signal can be expressed as

$$s(t) = A\cos\left(2\pi f_c t + \pi \sum_k a_k \int_{-\infty}^{t} g(\tau - kT)d\tau\right), \qquad (1)$$

where A is the signal amplitude, f_c is the carrier frequency, a_k is the input data, T is the bit interval and $g(t)$ is the frequency pulse that determines the overall spectral characteristics of the modulated signal. The bit interval for the coded system T_c is related to that for the uncoded system T by

$$T_c = rT, \qquad (2)$$

where r is the rate of the code.

For GMSK, $g(t)$ is given by [5]

$$g(t) = K\{\mathrm{erf}[cB_b(t + \frac{T}{2})] - \mathrm{erf}[cB_b(t - \frac{T}{2})]\}, \qquad (3)$$

where $\mathrm{erf}(\cdot)$ is the error function defined as

$$\mathrm{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u \mathrm{e}^{-y^2} dy, \qquad (4)$$

K is a constant chosen so that the pulse area is equal to 1/2, B_b is the $3dB$ bandwidth of the Gaussian prefilter used in GMSK modulators and c is a constant given by $c = \pi\sqrt{2/\ln 2}$. The frequency pulse is a continuous function that is truncated so that it is zero outside the interval $[-LT/2, LT/2]$. The parameter L is the memory length of the modulation, i.e., the number of symbols which a single symbol can affect. In this correspondence, we use $L = 3$. The spectrum of a GMSK signal, and hence the bandwidth, can be easily controlled by the parameter B_b. A larger value of B_b results in a larger signal bandwidth.

B. Bandwidth

In order to determine what combination of coding rate and B_b for the coded system results in the same bandwidth as the uncoded system, we must choose a measure of bandwidth. In this correspondence, we use the percent power containment bandwidth, denoted by B_x and defined as the bandwidth which contains $x\%$ of the signal power, e.g., $B_{99.9}$ is the bandwidth that contains 99.9% of the signal power. For GMSK, B_{90}, B_{99} and $B_{99.9}$ are plotted in Fig. 2.

The $B_{99.9}$ curve shows a sharp increase when B_bT is increased from 0.45 to 0.5. As can be seen from power spectral density plots of GMSK, e.g., [1], this increase is due to the enlargement of the main lobe and side lobes. Since $B_{99.9}$ is much more sensitive to changes in the power spectrum, a sharp increase appears only in this plot.

C. Error Rate

For transmission in an AWGN channel, the bit error rate performance of GMSK at high signal-to-noise ratio (SNR) values is given by

$$p = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{d_{min}^2 E}{2N_0}}\right), \qquad (5)$$

where $\mathrm{erfc}(\cdot)$ is the complementary error function defined as

$$\mathrm{erfc}(u) = 1 - \mathrm{erf}(u), \qquad (6)$$

d_{min} is the normalized minimum Euclidean distance between the signal representing “0” and the signal representing “1”, E is the energy per transmitted bit and $N_0/2$ is the power spectral density of the AWGN, i.e., the SNR is equal to E/N_0.

The normalized minimum Euclidean distance for GMSK is defined and plotted in [1]. As B_b is increased, d_{min}^2 approaches a maximum value of 2, which corresponds to the performance of

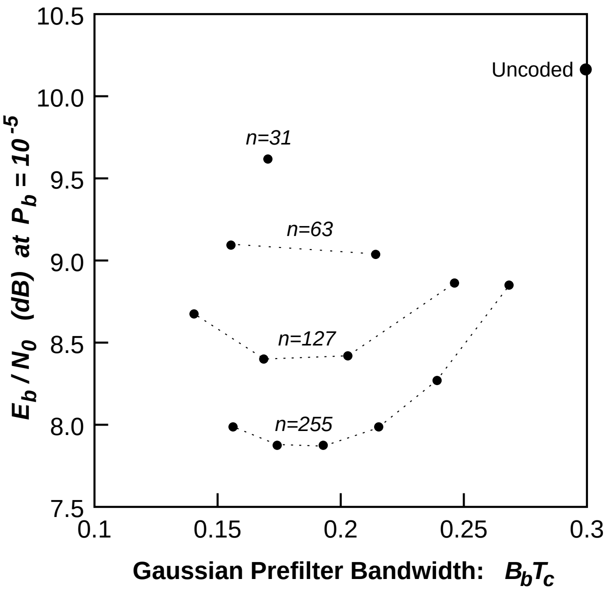

Fig. 3. The SNR required to achieve a bit error rate of 10^{-5} as a function of the normalized Gaussian prefilter bandwidth, B_bT_c, for various BCH codes with block length n. The total system bandwidth is equal to $B_{99.9}T = 1.167$, which corresponds to an uncoded system with $B_bT = 0.3$.

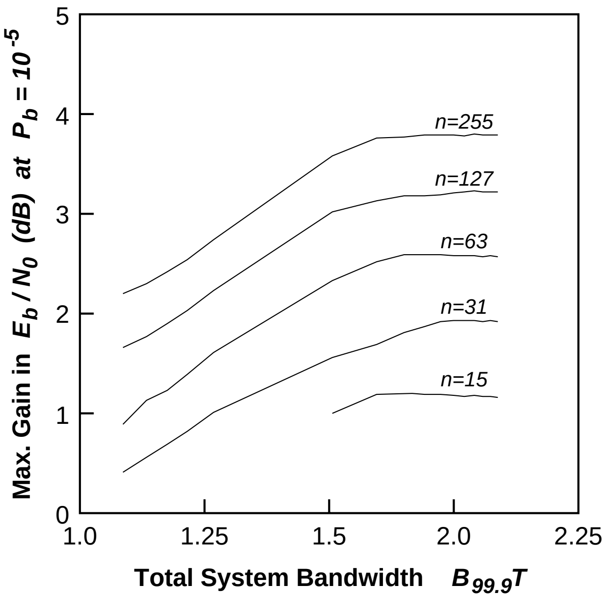

Fig. 4. The maximum gain, relative to the uncoded system, in SNR achieved at a bit error rate of 10^{-5} using BCH block codes as a function of the total system bandwidth for various block lengths n.

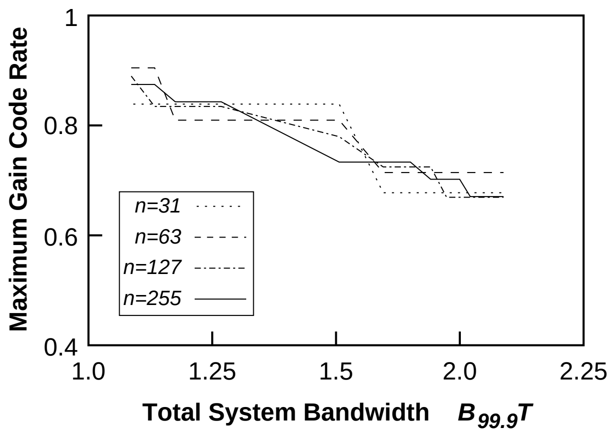

Fig. 5. The code rate at which the maximum gain is achieved for BCH codes as a function of the total system bandwidth for various block lengths n.

optimal antipodal signalling. Although GMSK is a modulation scheme with memory, in this correspondence we employ a simple demodulation scheme that only relies on the minimum distance between any two of several possible signal shapes in a bit interval.

To make a fair comparison between the uncoded and coded systems, the value of E used in (5) for the uncoded system is E_b, which is the energy per transmitted information bit. For the coded system the value of E used is rE_b, since the energy for the uncoded bits is spread among the more numerous coded bits.

III. Block Codes

A. Performance

Block codes are usually denoted by (n, k, t), where n is the block length, k is the number of information bits and t is the number of errors that can be corrected. The code rate is defined as k/n.

The approximate bit error rate when a block code is used in an AWGN channel is given by [9]

$$P_b = \frac{1}{n} \sum_{j=t+1}^{n} j \binom{n}{j} p^j (1-p)^{n-j}, \qquad (7)$$

where p is the error probability of the channel. For GMSK, p is given by (5).

B. Optimization Problem

For a given total transmission bandwidth B and SNR, what is the minimum probability of error that is possible using a coded GMSK system? Equivalently, what is the minimum SNR required to achieve a certain probability of error? This problem basically amounts to finding the best combination of GMSK modulation and code rate for a given bandwidth B. It is well known that the performance can be improved by increasing the block length of the code, so we consider this optimization problem for various block lengths.

Mathematically, the optimization problem can be stated as follows.

Minimize (7) subject to $B, n, E_b/N_0 = constant$

The parameter that can be varied during the optimization is t. Since this is a discrete optimization problem, i.e., the variable t can take only positive integer values, we obtain the solution by computing (7) for values of t starting from 1 until the minimum value is obtained.

We realize that there can be only one minimum value due to the monotonic nature of the equations involved. Intuitively, as t increases, the code rate decreases, which results in a decrease in the bandwidth available for modulation. This in turn results in a lower d_{min} in (5). Therefore p increases as t increases. In the calculation of (7), as t is increased, p increases, but fewer terms are summed. The net result is a temporary decrease in P_b until the effect of an increased p overwhelms the decrease in the number of terms summed. This results in a single minimum value for P_b.

TABLE I
GMSK AND BCH CODE PARAMETERS TO ACHIEVE MAXIMUM GAIN

System Bandwidth $B_{99.9}T$	GMSK and BCH code parameters B_bT, (n, k, t) $n = 31$	$n = 63$	$n = 127$	$n = 255$
1.11	0.148,(31,26,1)	0.183,(63,57,1)	0.175,(127,113,2)	0.166,(255,223,4)
1.17	0.171,(31,26,1)	0.214,(63,57,1)	0.169,(127,106,3)	0.193,(255,223,4)
1.22	0.194,(31,26,1)	0.175,(63,51,2)	0.191,(127,106,3)	0.197,(255,215,5)
1.27	0.219,(31,26,1)	0.197,(63,51,2)	0.216,(127,106,3)	0.223,(255,215,5)
1.34	0.259,(31,26,1)	0.230,(63,51,2)	0.255,(127,106,3)	0.264,(255,215,5)
1.63	0.457,(31,26,1)	0.445,(63,51,2)	0.408,(127,99,4)	0.329,(255,187,9)
1.74	0.313,(31,21,2)	0.378,(63,45,3)	0.395,(127,92,5)	0.417,(255,187,9)
1.81	0.359,(31,21,2)	0.431,(63,45,3)	0.442,(127,92,5)	0.448,(255,187,9)
1.86	0.394,(31,21,2)	0.449,(63,45,3)	0.453,(127,92,5)	0.439,(255,179,10)
1.90	0.428,(31,21,2)	0.455,(63,45,3)	0.411,(127,85,6)	0.450,(255,179,10)
1.94	0.441,(31,21,2)	0.459,(63,45,3)	0.432,(127,85,6)	0.455,(255,179,10)
1.96	0.448,(31,21,2)	0.462,(63,45,3)	0.442,(127,85,6)	0.443,(255,171,11)
1.99	0.452,(31,21,2)	0.465,(63,45,3)	0.448,(127,85,6)	0.450,(255,171,11)
2.01	0.455,(31,21,2)	0.468,(63,45,3)	0.452,(127,85,6)	0.453,(255,171,11)
2.03	0.457,(31,21,2)	0.467,(63,45,3)	0.454,(127,85,6)	0.455,(255,171,11)
2.05	0.460,(31,21,2)	0.472,(63,45,3)	0.457,(127,85,6)	0.457,(255,171,11)

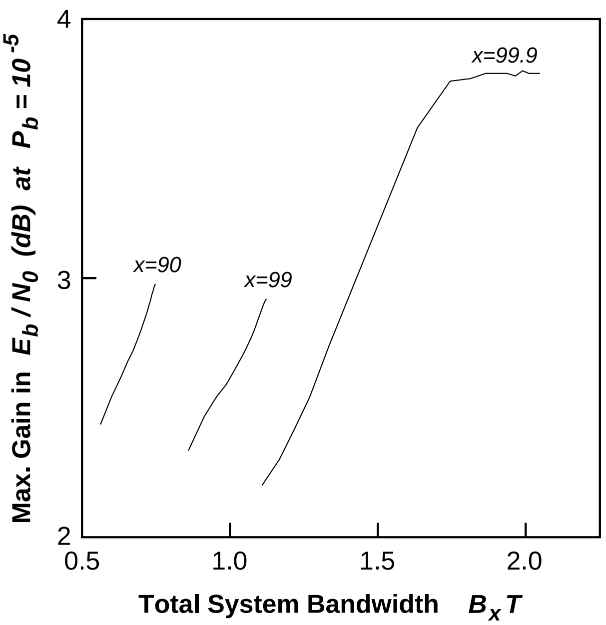

Fig. 6. The maximum gain, relative to the uncoded system, in SNR achieved at a bit error rate of 10^{-5} using BCH codes for $n = 255$. The bandwidth was measured using the 90%, 99% and 99.9% power containment bandwidths.

C. Solution for Bose-Chaudhuri-Hocquenghem (BCH) Codes

In order to investigate the performance limits of coded GMSK, we solve the optimization problem using BCH codes, whose parameters are taken from [10, Appendix E].

In order to compute (7) under the above constraints, we proceed as follows.

1. Set $t = 1$
2. Select the BCH codes parameters for t
3. The amount of bandwidth that can be used for modulation is $B_x = rB$
4. Compute the value of B_b corresponding to B_x
5. Compute the minimum distance for this value of B_b
6. Compute p using (5) with $E/N_0 = rE_b/N_0$
7. Compute P_b using (7)
8. Increment t by 1
9. Repeat steps 2 to 8 until a minimum value is found.

As an example of the calculation results, we show the SNR required to achieve $P_b = 10^{-5}$ as a function of the GMSK modulation parameter B_b in Fig. 3. As the figure shows, as t is increased, the performance improves until a minimum is reached and then worsens. The maximum improvement over the uncoded system is approximately $2.3dB$, when a block length of 255 is used. We can also see that the improvement in performance as the block length is doubled is approximately constant.

As expected, there is an optimal combination of coding and modulation for a given total transmission bandwidth. Examining the equations involved, as t is increased, the code rate that can be used decreases. This results in a smaller GMSK modulation bandwidth and hence a smaller d_{min}^2 and a higher p. A larger value of p increases P_b if t were constant. On the other hand, a larger value of t decreases P_b if p were constant. Since t and p both increase, the net result of increasing t depends on how much p increases in response to an increase in t. As long as p does not increase too much, the increase in t is enough to lower P_b. At some point, p increases too much to be countered by the increase in t and P_b becomes worse. This results in the minimum SNR value shown in Fig.3.

If we repeat the above calculations for various system bandwidths, a set of curves similar to Fig. 3 results. Looking at the maximum gain achieved over the uncoded system, we obtain the set of curves shown in Fig. 4, where the maximum gain that can be achieved for each block length is shown as a function of the total system bandwidth. As the figure shows, the gain that is obtained increases as the total system bandwidth is increased. This is to be expected, because with more bandwidth, we can use lower rate codes without sacrificing the performance of the modulation scheme. However, the gain starts to saturate around $B_{99.9}T = 2$, but this is only a temporary phenomenon. If the system band-

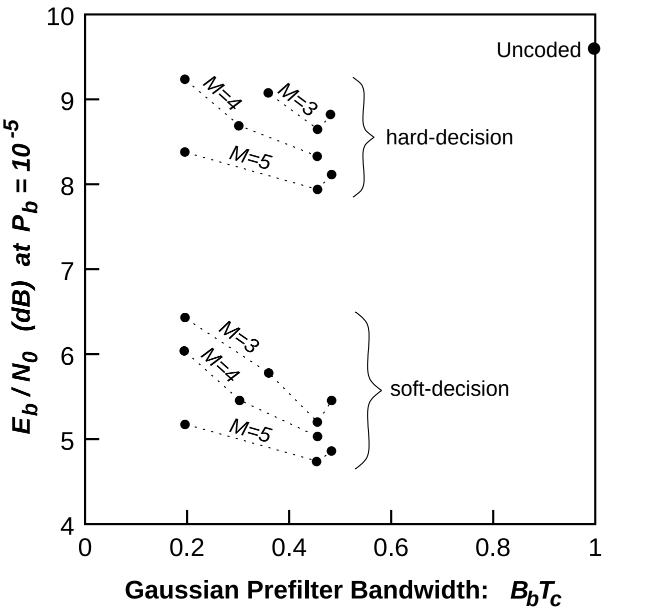

Fig. 7. The SNR required to achieve a bit error rate of 10^{-5} as a function of the normalized Gaussian prefilter bandwidth B_bT_c for various convolutional codes with M memory elements decoded using hard and soft decision Viterbi decoding. The total system bandwidth is equal to $B_{99.9}T = 2.05$, which corresponds to an uncoded system with $B_bT = 1.0$.

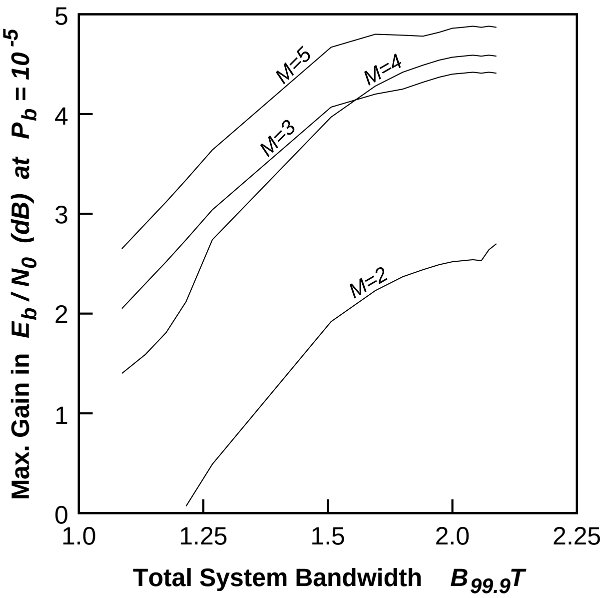

Fig. 8. The maximum gain in SNR achieved at a bit error rate of 10^{-5} for soft-decision decoded convolutional codes as a function of the total system bandwidth for various numbers of memory elements M.

width is increased further, the performance can be improved by using a lower rate code, because the performance of the modulation scheme alone is almost constant for $B_bT > 1$.

Now, if we look at the coding rate at which the maximum gain is achieved, i.e., the best performance of the coded GMSK system, we obtain the set of curves shown in Fig. 5. We can see that as the total system bandwidth is increased, the code rate at which the best performance is obtained decreases. There is some variation as the block length is varied, but this can be attributed to the lack of codes for short block lengths.

Fig. 5 also shows us the optimal balance between coding and modulation as a function of the system bandwidth. For example, when the total system bandwidth is $B_{99.9}T = 2$, the code rate at which the best performance is achieved is approximately 2/3. This means that using 2/3 of the bandwidth for modulation and the remaining 1/3 of the bandwidth for coding is the optimal balance between modulation and coding for GMSK with this system bandwidth. We can also see that more bandwidth should be used for modulation when the system bandwidth is small. On the other hand, when the system bandwidth is large, more of the bandwidth should be used for coding.

For a given system bandwidth, the best BCH code and GMSK modulation combination to use depends on several factors such as bandwidth measure and block code length. For the BCH codes which achieve the maximum gain using $B_{99.9}$, i.e., those in Fig. 5, the modulation and code parameters are shown in Table I.

In the previous discussion, we have used $B_{99.9}$ as the bandwidth measure. We now look at the effect of changing the bandwidth measure to a more conservative one. In Fig. 6 the maximum gain is shown when B_{90}, B_{99} and $B_{99.9}$ are used. In all cases, some coding gain can be achieved. However, not as much coding gain can be achieved using the 90% and 99% bandwidth measures. This is due to the fact that these bandwidth measures are less sensitive to changes in B_b as can be seen in Fig. 2. Therefore, in order to be able to use coding in these cases, B_b must be reduced more than when using the 99.9% bandwidth measure.

IV. Convolutional Codes

A. Code Definitions

Next, we consider coding of GMSK with convolutional codes. Convolutional codes are usually denoted by their code rate $r = k/n$, constraint length K, total number of memory elements M, and free distance d_{free}. For convolutional codes, k and n are the number of inputs and outputs, respectively, in the encoder. Here we use the definition for constraint length given in [10, p.268], i.e., the constraint length is the maximum number of bits in a single output stream that can be affected by any input bit.

The convolutional codes that we use are those given in [10, p.285] and [11, p.496], which have the largest d_{free} for a given K and M. These codes provide us with the best performance for convolutionally encoded GMSK systems. We use rates equal to 1/2, 4/7, 3/5, 2/3, 3/4 and 4/5, but codes are not tabulated for all values of K.

In order to use codes that are in some sense equivalent, we group the convolutional codes into sets according to the value of M. This means that all the codes in each set have the same number of states in the decoder trellis and hence the same decoding complexity.

B. Performance

The bit error rate performance of convolutional codes can be approximated by [10]

$$P_b = \frac{1}{k} b_{d_{free}} Z(d_{free}), \tag{8}$$

where $b_{d_{free}}$ is the total number of information bits associated with code sequences of weight d_{free} and $Z(d_{free})$ is a function of the channel and decoding method. For AWGN channels and hard-decision decoding, i.e., binary symmetric channels with crossover probability p, $Z(d_{free})$ is given by

$$Z(d_{free}) = \left[2\sqrt{p(1-p)}\right]^{d_{free}}. \tag{9}$$

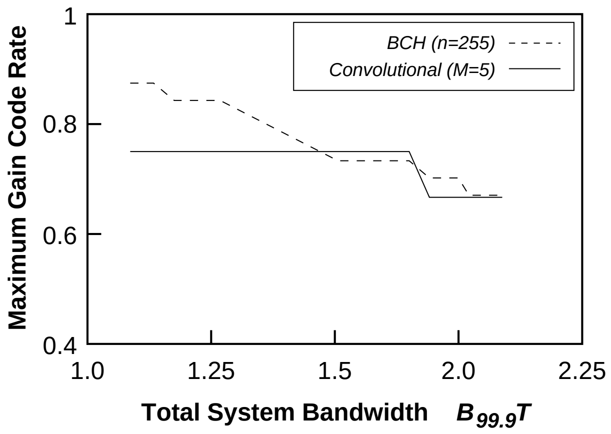

Fig. 9. The code rate at which the maximum gain is achieved as a function of the total system bandwidth for different types of codes.

TABLE II
GMSK AND CONVOLUTIONAL CODE PARAMETERS TO ACHIEVE MAXIMUM GAIN

System Bandwidth $B_{99.9}T$	GMSK and convolutional code parameters $B_bT, (r, d_{free})$			
	$M = 2$	$M = 3$	$M = 4$	$M = 5$
1.11	-	0.111,(3/4,4)	0.131,(4/5,3)	0.111,(3/4,5)
1.17	-	0.127,(3/4,4)	0.150,(4/5,3)	0.127,(3/4,5)
1.22	0.104,(2/3,3)	0.142,(3/4,4)	0.104,(2/3,5)	0.142,(3/4,5)
1.27	0.116,(2/3,3)	0.158,(3/4,4)	0.116,(2/3,5)	0.158,(3/4,5)
1.34	0.132,(2/3,3)	0.182,(3/4,4)	0.132,(2/3,5)	0.182,(3/4,5)
1.63	0.236,(2/3,3)	0.356,(3/4,4)	0.236,(2/3,5)	0.356,(3/4,5)
1.74	0.296,(2/3,3)	0.439,(3/4,4)	0.296,(2/3,5)	0.439,(3/4,5)
1.81	0.340,(2/3,3)	0.340,(2/3,4)	0.340,(2/3,5)	0.455,(3/4,5)
1.86	0.375,(2/3,3)	0.375,(2/3,4)	0.375,(2/3,5)	0.461,(3/4,5)
1.90	0.401,(2/3,3)	0.401,(2/3,4)	0.401,(2/3,5)	0.401,(2/3,6)
1.94	0.429,(2/3,3)	0.429,(2/3,4)	0.429,(2/3,5)	0.429,(2/3,6)
1.96	0.440,(2/3,3)	0.440,(2/3,4)	0.440,(2/3,5)	0.440,(2/3,6)
1.99	0.447,(2/3,3)	0.447,(2/3,4)	0.447,(2/3,5)	0.447,(2/3,6)
2.01	0.184,(1/2,5)	0.451,(2/3,4)	0.451,(2/3,5)	0.451,(2/3,6)
2.03	0.190,(1/2,5)	0.453,(2/3,4)	0.453,(2/3,5)	0.453,(2/3,6)
2.05	0.195,(1/2,5)	0.456,(2/3,4)	0.456,(2/3,5)	0.456,(2/3,6)

For soft-decision decoding in AWGN channels,

$$Z(d_{free}) = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{d_{min}^2 d_{free} E}{2N_0}}\right), \tag{10}$$

where d_{min}, E and N_0 are as in (5).

As with block codes, we calculate the value of SNR required to achieve a bit error rate of 10^{-5} for both hard and soft-decision Viterbi decoding. The results when the total system bandwidth is 2.05 is shown in Fig. 7. From this figure, we can see that the performance improves in relatively constant intervals as the total number of encoder memory elements, M, is increased. We can also see that soft-decision decoding is roughly $3dB$ better than hard-decision decoding. As was the case for block codes, the performance has an optimal value due to the tradeoff in modulation performance and coding gain.

If we focus on soft-decision decoding and compute the performance gain over the uncoded system as a function of the total system bandwidth, we obtain the results shown in Fig. 8. As was the case for block codes, we see that the gain in performance improves as the total system bandwidth is increased.

C. Comparison to Block Codes

Looking at the code rate which achieves the maximum gain in performance, we find that similar results are obtained for convolutional and block codes. As shown in Fig. 9, the optimal code rates decrease as the system bandwidth increases. Due to the small number of codes and corresponding rates, the curve for convolutional codes is lower at smaller system bandwidth values, but the trend and code rates are comparable to the block codes. This shows that the best tradeoff between modulation and coding is approximately independent of the type of code used.

The parameters for the best convolutional code and GMSK modulation combinations are shown in Table II.

V. CONCLUSIONS

In this correspondence, optimization of coded Gaussian Minimum Shift Keying was examined. The optimal code rate was found as a function of the total system bandwidth. From this result, it was found that the percentage of the system bandwidth that should be used for coding is small when the system bandwidth is small and large when the system bandwidth is large. For a total system bandwidth equal to twice the bit rate, it was found that 1/3 of the system bandwidth should be used for coding and 2/3 for modulation. Using BCH and convolutional codes, it was found that the bit error rate performance can be easily improved by more than $3dB$ over an uncoded system with the same bandwidth. Optimization in fading channels is currently under examination.

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